

THE NON-ABELIAN TENSOR SQUARE OF RESIDUALLY FINITE GROUPS

R. BASTOS AND N. R. ROCCO

ABSTRACT. Let m, n be positive integers and p a prime. We denote by $\nu(G)$ an extension of the non-abelian tensor square $G \otimes G$ by $G \times G$. We prove that if G is a residually finite group satisfying some non-trivial identity $f \equiv 1$ and for every $x, y \in G$ there exists a p -power $q = q(x, y)$ such that $[x, y^\varphi]^q = 1$, then the derived subgroup $\nu(G)'$ is locally finite (Theorem A). Moreover, we show that if G is a residually finite group in which for every $x, y \in G$ there exists a p -power $q = q(x, y)$ dividing p^m such that $[x, y^\varphi]^q$ is left n -Engel, then the non-abelian tensor square $G \otimes G$ is locally virtually nilpotent (Theorem B).

1. INTRODUCTION

The non-abelian tensor square $G \otimes G$ of a group G as introduced by Brown and Loday [BL84, BL87] is defined to be the group generated by all symbols $g \otimes h$, $g, h \in G$, subject to the relations

$$gg_1 \otimes h = (g^{g_1} \otimes h^{g_1})(g_1 \otimes h) \quad \text{and} \quad g \otimes hh_1 = (g \otimes h_1)(g^{h_1} \otimes h^{h_1})$$

for all $g, g_1, h, h_1 \in G$.

This is a particular case of a non-abelian tensor product $G \otimes H$, of groups G and H , under the assumption that G and H acts one on each other and on itself by conjugation.

We observe that the defining relations of the tensor square can be viewed as abstractions of commutator relations; thus in [Roc91] the following related construction is considered: Let G and H be groups and $\varphi : H \rightarrow H^\varphi$ an isomorphism (H^φ is an isomorphic copy of H , where $h \mapsto h^\varphi$, for all $h \in H$). Define the group $\nu(G)$ to be

$$\nu(G) := \langle G, G^\varphi \mid [g_1, g_2^\varphi]^{g_3} = [g_1^{g_3}, (g_2^{g_3})^\varphi] = [g_1, g_2^\varphi]^{g_3^\varphi}, \quad g_i \in G \rangle.$$

The motivation for studying $\nu(G)$ is the commutator connection:

PROPOSITION 1.1. ([Roc91, Proposition 2.6]) *The map $\Phi : G \otimes G \rightarrow [G, G^\varphi]$, defined by $g \otimes h \mapsto [g, h^\varphi]$, $\forall g, h \in G$, is an isomorphism.*

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From now on we identify the tensor square $G \otimes G$ with the subgroup $[G, G^\varphi]$ of $\nu(G)$ (see for instance [NR12] for more details).

It is a natural task to study structural aspects of $\nu(G)$ and $G \otimes G$ for different classes of infinite groups G . Moravec [Mor08] proves that if G is locally finite then so is $G \otimes G$ (and, consequently, also $\nu(G)$). In the article [BR] the authors establish some structural results concerning $\nu(G)$ when G is a finite-by-nilpotent group. In the present paper we are concerned with the non-abelian tensor square of residually finite groups.

According to the solution of the Restricted Burnside Problem (Zelmanov, [Zel91a, Zel91b]) every residually finite group of finite exponent is locally finite. Another interesting result in this context, due to Shumyatsky [Shu05] states that if G is a residually finite group satisfying a non-trivial identity and generated by a normal commutator-closed set of p -elements, then G is locally finite. A subset X of a group G is called *commutator-closed* if $[x, y] \in X$ whenever $x, y \in X$. In a certain way, our results can be viewed as generalizations of some of the above results.

Theorem A. *Let p a prime. Let m be a positive integer and G a residually finite group satisfying some non-trivial identity $f \equiv 1$. Suppose that for every $x, y \in G$ there exists a positive integer $q = q(x, y)$ dividing p^m such that the tensor $[x, y^\varphi]^q = 1$. Then the derived subgroup $\nu(G)'$ is locally finite.*

Let G be a group. For elements x, y of G we define $[x, {}_1y] = [x, y]$ and $[x, {}_{i+1}y] = [[x, {}_iy], y]$ for $i \geq 1$, where $[x, y] = x^{-1}y^{-1}xy$. An element $y \in G$ is called *left n -Engel* if for any $x \in G$ we have $[x, {}_ny] = 1$. The group G is called a *left n -Engel group* if $[x, {}_ny] = 1$ for all $x, y \in G$.

Another result that was deduced following the solution of the Restricted Burnside Problem is that any residually finite n -Engel group is locally nilpotent (J.S. Wilson, [Wil91]). Later, Shumyatsky proved that if G is a residually finite group in which every commutator is left n -Engel in G , then the derived subgroup G' is locally nilpotent [Shu99b, Shu00]. In the present paper we establish the following related result.

Theorem B. *Let m, n be positive integers and p a prime. Suppose that G is a residually finite group in which for every $x, y \in G$ there exists a positive integer $q = q(x, y)$ dividing p^m such that the element $[x, y^\varphi]^q$ is left n -Engel in $\nu(G)$. Then the non-abelian tensor square $G \otimes G$ is locally virtually nilpotent.*

A natural question arising in the context of Theorem B is whether the theorem remains valid when q is allowed to be an arbitrary positive integer, rather than a p -power (see Proposition 4.4 in Section 4 for details).

The paper is organized as follows. Section 2 presents terminology and some preparatory results that are later used in the proofs of our main results. In Section 3 we describe some important ingredients of what is often called Lie methods in group theory. These are essential in the proof of Theorem B. Section 4 contains the proofs of the main theorems.

2. PRELIMINARY RESULTS

The following basic properties are consequences of the defining relations of $\nu(G)$ and the commutator rules (see [Roc91, Section 2] for more details).

LEMMA 2.1. *The following relations hold in $\nu(G)$, for all $g, h, x, y \in G$.*

- (i) $[g, h^\varphi]^{[x, y^\varphi]} = [g, h^\varphi]^{[x, y]}$;
- (ii) $[g, h^\varphi, x^\varphi] = [g, h, x^\varphi] = [g, h^\varphi, x] = [g^\varphi, h, x^\varphi] = [g^\varphi, h^\varphi, x] = [g^\varphi, h, x]$;
- (iii) *If $h \in G'$ (or if $g \in G'$) then $[g, h^\varphi][h, g^\varphi] = 1$;*
- (iv) $[g, [h, x]^\varphi] = [[h, x], g^\varphi]^{-1}$;
- (v) $[[g, h^\varphi], [x, y^\varphi]] = [[g, h], [x, y]^\varphi]$.

LEMMA 2.2. *Let G be a group. Then $X = \{[a, b^\varphi] \mid a, b \in G\}$ is a normal commutator-closed subset of $\nu(G)$.*

Proof. By definition, $[a, b^\varphi]^{y^\varphi} = [a, b^\varphi]^y = [a^y, (b^y)^\varphi]$, for every $a, b, y \in G$. It follows that X is a normal subset of $\nu(G)$. Now, it remains to prove that X is a commutator-closed subset of $\nu(G)$. Choose arbitrarily elements $a, b, c, d \in G$ and consider the elements $[a, b^\varphi], [c, d^\varphi] \in X$. By Lemma 2.1 (v),

$$[[a, b^\varphi], [c, d^\varphi]] = [[a, b], [c, d]^\varphi] \in X.$$

The result follows. \square

The epimorphism $\rho : \nu(G) \rightarrow G$, given by $g \mapsto g, h^\varphi \mapsto h$, induces the derived map

$$\rho' : [G, G^\varphi] \rightarrow G',$$

$[g, h^\varphi] \mapsto [g, h]$ for all $g, h \in G$. In the notation of [Roc94, Section 2], let $\mu(G)$ denote the kernel of ρ' . In particular,

$$\frac{[G, G^\varphi]}{\mu(G)} \cong G'.$$

The next lemma is a particular case of Theorem 3.3 in [Roc91].

LEMMA 2.3. *Let G be a group. Then the derived subgroup*

$$\nu(G)' = ([G, G^\varphi] \cdot G') \cdot (G')^\varphi,$$

where “ \cdot ” denotes an internal semi-direct product.

LEMMA 2.4. *Let m, n be positive integers and x, y be elements in a group G . If $[x, y^\varphi]^m$ is left n -Engel element in $\nu(G)$, then the power $[x, y]^m$ is left n -Engel in G .*

Proof. By assumption, $[z, {}_n[x, y^\varphi]^m] = 1$ for every $z \in G$. It follows that

$$1 = \rho([z, {}_n[x, y^\varphi]^m]) = [z, {}_n[x, y]^m],$$

for every $z \in G$, i.e., $[x, y]^m$ is left n -Engel in G , and the lemma follows. \square

The remainder of this section will be devoted to describe certain finiteness conditions for residually finite and locally graded groups. Recall that a group is *locally graded* if every non-trivial finitely generated subgroup has a proper subgroup of finite index. Interesting classes of groups (e.g. locally finite groups, locally nilpotent groups, residually finite groups) are locally graded.

We need the following result, due to Shumyatsky [Shu05].

THEOREM 2.5. *Let G be a residually finite group satisfying some non-trivial identity $f \equiv 1$. Suppose G is generated by a normal commutator-closed set X of p -elements. Then G is locally finite.*

Next, we extend this result to the class of locally graded groups.

LEMMA 2.6. *Let p be a prime. Let G be a locally graded group satisfying some non-trivial identity $f \equiv 1$. Suppose that G is generated by finitely many elements of finite order and for every $x, y \in G$ there exists a p -power $q = q(x, y)$ such that $[x, y]^q = 1$. Then G is finite.*

Proof. Let R be the finite residual of G , i.e., the intersection of all subgroups of finite index in G . If $R = 1$, then G is residually finite. Since the set of all commutators is a normal commutator-closed subset of p -elements, we have G' is locally finite (Theorem 2.5). Moreover, G/G' is generated by finitely many elements of finite order. It follows that G/G' is finite and so G' is finitely generated. Consequently, G is finite. So, we can assume that $R \neq 1$. By the same argument, G/R is finite and thus R is finitely generated. As R is locally graded we have that R contains a proper subgroup of finite index in G , which gives a contradiction. \square

It is easy to see that a quotient of a residually finite group need not be residually finite (see for instance [Rob96, 6.1.9]). This includes the class of locally graded groups. However, the next result gives a sufficient condition for a quotient to be locally graded.

LEMMA 2.7. (*Longobardi, Maj, Smith, [LMS95]*) *Let G be a locally graded group and N a normal locally nilpotent subgroup of G . Then G/N is locally graded.*

3. ON LIE ALGEBRAS ASSOCIATED WITH GROUPS

Let L be a Lie algebra over a field \mathbb{K} . We use the left normed notation: thus if l_1, l_2, \dots, l_n are elements of L , then

$$[l_1, l_2, \dots, l_n] = [\dots [[l_1, l_2], l_3], \dots, l_n].$$

We recall that an element $a \in L$ is called *ad-nilpotent* if there exists a positive integer n such that $[x, {}_n a] = 0$ for all $x \in L$. When n is the least integer with the above property then we say that a is ad-nilpotent of index n .

Let $X \subseteq L$ be any subset of L . By a commutator of elements in X , we mean any element of L that could be obtained from elements of X by means of repeated operation of commutation with an arbitrary system of brackets including the elements of X . Denote by F the free Lie algebra over \mathbb{K} on countably many free generators x_1, x_2, \dots . Let $f = f(x_1, x_2, \dots, x_n)$ be a non-zero element of F . The algebra L is said to satisfy the identity $f = 0$ if $f(l_1, l_2, \dots, l_n) = 0$ for any $l_1, l_2, \dots, l_n \in L$. In this case we say that L is PI. Now, we recall an important theorem of Zelmanov [Zel90, Theorem 3] that has many applications in group Theory.

THEOREM 3.1. *Let L be a Lie algebra generated by l_1, l_2, \dots, l_m . Assume that L is PI and that each commutator in the generators is ad-nilpotent. Then L is nilpotent.*

Let G be a group and p a prime. In what follows,

$$D_i = D_i(G) = \prod_{jp^k \geq i} (\gamma_j(G))^{p^k}$$

denotes the i -th dimension subgroup of G in characteristic p . These subgroups form a central series of G known as the *Zassenhaus-Jennings-Lazard series* (this can be found in Shumyatsky [Shu00, Section 2]). Set $L(G) = \bigoplus D_i/D_{i+1}$. Then $L(G)$ can naturally be viewed as a Lie algebra over the field \mathbb{F}_p with p elements. The subalgebra of $L(G)$ generated by D_1/D_2 will be denoted by $L_p(G)$. The nilpotency of $L_p(G)$

has strong influence in the structure of a finitely generated group G . The following result is due to Lazard [Laz65].

THEOREM 3.2. *Let G be a finitely generated pro- p group. If $L_p(G)$ is nilpotent, then G is p -adic analytic.*

Let $x \in G$, and let $i = i(x)$ be the largest integer such that $x \in D_i$. We denote by \tilde{x} the element $xD_{i+1} \in L(G)$. Now, we can state one condition for \tilde{x} to be ad-nilpotent.

LEMMA 3.3. (Lazard, [Laz54, page 131]) *For any $x \in G$ we have $(ad \tilde{x})^q = ad(\tilde{x}^q)$.*

REMARK 3.4. *We note that q in Lemma 3.3 does not need to be a p -power. In fact, is easy to see that if p^s is the maximal p -power dividing q , then \tilde{x} is ad-nilpotent of index at most p^s .*

The following result is an immediate corollary of [WZ92, Theorem 1].

LEMMA 3.5. *Let G be any group satisfying a group-law. Then $L(G)$ is PI.*

For a deeper discussion of applications of Lie methods to group theory we refer the reader to [Shu00].

4. PROOF OF THE MAIN RESULTS

By a well-known theorem of Schmidt [Rob96, 14.3.1], the class of locally finite groups is closed with respect to extension. By Lemma 2.3, $\nu(G)' = ([G, G^\varphi] \cdot G') \cdot (G')^\varphi$. From this and Proposition 1.1 we conclude that the non-abelian tensor square $G \otimes G$ is locally finite if and only if $\nu(G)'$ is locally finite. The next result will be needed in the proof of Theorem A.

PROPOSITION 4.1. *Let G be a group in which G' is locally finite. The following properties are equivalent.*

- (a) $[G, G^\varphi]$ is locally finite;
- (b) $\nu(G)'$ is locally finite;
- (c) For every $x, y \in G$ there exists a positive integer $m = m(x, y)$ such that $[x, y^\varphi]^m = 1$.

Proof. (a) \Rightarrow (b): It is sufficient to see that $\nu(G)' = ([G, G^\varphi] \cdot G') \cdot (G')^\varphi$ (Theorem 2.3).

(b) \Rightarrow (c): Since $\nu(G)'$ is locally finite, it follows that every element in $\nu(G)'$ has finite order. In particular, for every $x, y \in G$ the element

$[x, y^\varphi]$ has finite order.

(c) \Rightarrow (a): Let us first prove that $[G, G^\varphi]$ is locally finite. Let W be a finitely generated subgroup of $[G, G^\varphi]$. Since the factor group $[G, G^\varphi]/\mu(G)$ is isomorphic to G' , it follows that W is central-by-finite. By Schur's Lemma [Rob96, 10.1.4], the derived subgroup W' is finite. Since W/W' is an abelian group generated by finite many elements of finite order, it follows that W/W' is also finite. Thus $[G, G^\varphi]$ is locally finite. Now, Lemma 2.3 shows that $\nu(G)' = ([G, G^\varphi] \cdot G') \cdot (G')^\varphi$. Therefore $\nu(G)'$ is locally finite, by [Rob96, 14.3.1]. \square

We are now in a position to prove Theorem A.

Theorem A. *Let p be a prime and G a residually finite group satisfying some non-trivial identity $f \equiv 1$. Suppose that for every $x, y \in G$ there exists a p -power $q = q(x, y)$ such that $[x, y^\varphi]^q = 1$. Then the derived subgroup $\nu(G)'$ is locally finite.*

Proof. The proof is completed by showing that G' is locally finite (Proposition 4.1). Set

$$X = \{ [a, b^\varphi]; a, b \in G \}.$$

By Lemma 2.2, X is a normal commutator-closed subset of p -elements in $\nu(G)$. Since $[G, G^\varphi]/\mu(G) \cong G'$, we conclude that the derived subgroup G' is residually finite satisfying the identity $f \equiv 1$. In the same manner we can see that for every x, y there exists a p -power $q = q(x, y)$ such that $[x, y]^q = 1$. Theorem 2.5 now implies shows that G' is locally finite. The proof is complete. \square

REMARK 4.2. *For more details concerning residually finite groups in which the derived subgroup is locally finite see [Shu99a, Shu05].*

Now we will deal with Theorem B: *Let m, n be positive integers and p a prime. Suppose that G is a residually finite group in which for every $x, y \in G$ there exists a p -power $q = q(x, y)$ dividing p^m such that $[x, y^\varphi]^q$ is left n -Engel in $\nu(G)$. Then the non-abelian tensor square $G \otimes G$ is locally virtually nilpotent.*

We denote by \mathcal{N} the class of all finite nilpotent groups. The following result is a straightforward corollary of [Wil91, Lemma 2.1].

LEMMA 4.3. *Let G be a finitely generated residually- \mathcal{N} group. For each prime p , let R_p be the intersection of all normal subgroups of G of finite p -power index. If G/R_p is nilpotent for each p , then G is nilpotent.*

The next result will be helpful in the proof of Theorem B.

PROPOSITION 4.4. *Let m, n be positive integers and p a prime. Suppose that G is a residually finite group in which for every $x, y \in G$ there exists a p -power $q = q(x, y)$ dividing p^m such that $[x, y]^q$ is left n -Engel. Then $(G')^{p^m}$ is locally nilpotent. Moreover, the derived subgroup G' is locally virtually nilpotent.*

Proof. Let K be the derived subgroup of G . Choose an arbitrary commutator $x \in G$ and let $q = q(x)$ be a positive integer $q = q(x)$ dividing p^m such that x^q is left n -Engel. It suffices to prove that the normal closure of x^q in G , $\langle (x^q)^h \mid h \in G \rangle$, is locally nilpotent. In particular, $x^q \in HP(G)$. Let b_1, \dots, b_t be finitely many elements in G . Let $h_i = (x^q)^{b_i}$, $i = 1, \dots, t$ and $H = \langle h_1, \dots, h_t \rangle$. We only need to show that H is soluble (Gruenberg's Theorem [Rob96, 12.3.3]). As a consequence of Lemma 4.3, we can assume that H is residually- p for some prime p . Let $L = L_p(H)$ be the Lie algebra associated with the Zassenhaus-Jennings-Lazard series

$$H = D_1 \geq D_2 \geq \dots$$

of H . Then L is generated by $\tilde{h}_i = h_i D_2$, $i = 1, 2, \dots, t$. Let \tilde{h} be any Lie-commutator in \tilde{h}_i and h be the group-commutator in h_i having the same system of brackets as \tilde{h} . Since for any group commutator h in $h_1 \dots, h_t$ there exists a positive integer q dividing p^m such that h^q is left n -Engel, Lemma 3.3 shows that any Lie commutator in $\tilde{h}_1 \dots, \tilde{h}_t$ is ad-nilpotent. On the other hand, for every commutator $x = [a_1, b_1]$ there exists a positive integer $q = q(x)$ dividing p^m such that x^q is left n -Engel, then G satisfies the identity

$$f = [z, {}_n[x_1, y_1], {}_n[x_1, y_1]^p, {}_n[x_1, y_1]^{p^2}, {}_n[x_1, y_1]^{p^3}, \dots, {}_n[x_1, y_1]^{p^m}] \equiv 1$$

and therefore, by Lemma 3.5, L satisfies some non-trivial polynomial identity. Now Zelmanov's Theorem 3.1 implies that L is nilpotent. Let \hat{H} denote the pro- p completion of H . Then $L_p(\hat{H}) = L$ is nilpotent and \hat{H} is a p -adic analytic group by Theorem 3.2. Clearly H cannot have a free subgroup of rank 2 and so, by Tits' Alternative [Tit72], H is virtually soluble. As H is residually- p we have H is soluble.

Choose arbitrarily finitely many commutators $[a_1, b_1], \dots, [a_t, b_t]$ and let $M = \langle [a_1, b_1], \dots, [a_t, b_t] \rangle$. Since MK^{p^m} is residually finite and K^{p^m} is locally graded, we conclude that $(MK^{p^m})/K^{p^m}$ is also locally graded (Lemma 2.7). Now, looking at the quotient $(MK^{p^m})/K^{p^m}$, the claim is immediate from Lemma 2.6. As $M/(M \cap K^{p^m})$ is finite we have $M \cap K^{p^m}$ finitely generated. In particular, $M \cap K^{p^m}$ is nilpotent and M is virtually nilpotent. This completes the proof. \square

REMARK 4.5. *The above proposition is no longer valid if the assumption of residual finiteness of G is dropped. In [DK99] G. S. Deryabina and P. A. Kozhevnikov showed that there exists an integer $N \geq 1$ such that for every odd number $n > N$ there is a group G with commutator subgroup G' not locally finite and satisfying the identity*

$$f = [x, y]^n \equiv 1.$$

In particular, G' cannot be locally virtually nilpotent. For more details concerning groups in which certain powers are left Engel elements see [Bas15, BS15, BSTT13, STT16].

The proof of Theorem B is now easy.

Proof of Theorem B. Let M be a finitely generated subgroup of $[G, G^\varphi]$. Clearly, there exist finitely many elements $a_1, \dots, a_s, b_1, \dots, b_s \in G$ such that

$$M \leq \langle [a_i, b_i^\varphi] \mid i = 1, \dots, s \rangle = H.$$

It suffices to prove that H is virtually nilpotent. Since for every $a, b \in G$ there exists a p -power $q = q(a, b)$ dividing p^m such that $[a, b^\varphi]^q$ is left n -Engel, we deduce that the power $[a, b]^q$ is left n -Engel (Lemma 2.4). That G' is locally virtually nilpotent follows from Proposition 4.4. Therefore, since $\mu(G)$ is the kernel of the derived map ρ' , it follows that $H/(H \cap \mu(G))$ is virtually nilpotent. As $\mu(G)$ is a central subgroup of $\nu(G)$ we have H virtually nilpotent, which proves the theorem. \square

We conclude this paper by formulating an open problem related to the results described here.

Problem 1. *Let m, n be positive integers. Suppose that G is a residually finite group in which for every $x, y \in G$ there exists a positive integer $q = q(x, y)$ dividing m such that $[x, y^\varphi]^q$ is left n -Engel in $\nu(G)$. Is it true that $G \otimes G$ is locally virtually nilpotent?*

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DEPARTAMENTO DE MATEMATÁTICA, UNIVERSIDADE DE BRASÍLIA, BRASILIA-
DF, 70910-900 BRAZIL

E-mail address: bastos@mat.unb.br

DEPARTAMENTO DE MATEMATÁTICA, UNIVERSIDADE DE BRASÍLIA, BRASILIA-
DF, 70910-900 BRAZIL

E-mail address: norai@unb.br